A HIERARCHICAL SEGMENTATION ALGORITHM BASED ON HEPTA-TREE

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ABSTRACT
This paper is concerned with a Hierarchical Markov Random Field (HMRF) algorithm for image segmentation, based on samples belonging to a hexagonal grid. Most of image segmentation algorithms use the topology based on the classical \( \mathbb{Z}^2 \) grid, i.e., the squared grid, because this is an extension from the one-dimensional case. Nevertheless, the \( \mathbb{Z}^2 \) grid is not optimal according to the Shannon sampling theorem: the optimal one for image sampling is the hexagonal grid \([16, 1]\). In this paper, we adapt to hexagonal topology a hierarchical image segmentation algorithm developed previously on a \( \mathbb{Z}^2 \) grid. We present here a new structure, called the hepta-tree, adapted to hexagonal grids. Unsupervised segmentation results are compared on synthetic images issued from the both sampling grids.

**Keywords:** image, hidden Markov tree, quad-tree, hepta-tree, hexagonal grid, unsupervised Bayesian segmentation.

1 INTRODUCTION

This paper presents an original unsupervised segmentation method based on hexagonal grid. Most of image segmentation algorithms suppose that the image is sampled on a squared grid (\( \mathbb{Z}^2 \)). Previous works [16, 1] have demonstrated that this grid is not optimal according to the Shannon sampling theorem. Moreover, the description of the neighborhood of a pixel is ambiguous (we can choose 4-connexity neighborhood or 8-connexity neighborhood). The hexagonal grid, denoted \( A_2 \), allows to sample images in an optimal way, in the sense of the Shannon sampling theorem. The second section of this paper will describe this new topology and illustrates its optimality features. In the following, we present an efficient unsupervised HMRF segmentation algorithm for images sampled on a \( \mathbb{Z}^2 \) grid. This approach is based on a hierarchical structure: the quad-tree. We present a generalization of this way to hierarchical tree adapted for an image sampled on a \( A_2 \) grid: the hepta-tree. Then section 3, describes the different steps of the unsupervised algorithm obtained on the hepta-tree structure, whereas results on synthetic images are reported in section 4, and are compared with results based on a \( \mathbb{Z}^2 \) grid. Then the last section contains perspectives and concluding remarks.

2 THE HEPTA-TREE STRUCTURE

2.1 The hexagonal grid sampling

Images we are working on, come from the sampling of continuous real data. Usually, a mono-dimensional signal is sampled along the \( \mathbb{Z} \) axis. That is why, in image processing, an extension of the 1D case consists in sampling images on a \( \mathbb{Z}^2 \) grid (cf. Figure 1a).

![Figure 1: a) squared grid: \( \mathbb{Z}^2 \). b) hexagonal grid: \( A_2 \).](image)

Although this sampling seems to be obvious, it is not optimal in the sense of the Shannon sampling theorem. Another topology exists, based on the hexagonal grid denoted \( A_2 \) (cf. Figure 1b). On this grid, a pixel has 6 neighbors, distant from the same length. With a \( \mathbb{Z}^2 \) grid, the definition of a neighborhood is more ambiguous. In the 8-connexity case, 4 neighbors are distant from the length \( D \), and 4 others are distant from \( D \sqrt{2} \).

The optimality of the \( A_2 \) grid can be illustrated by viewing the spectrum of an image. We suppose that our images have an isotropic spectrum, i.e., the energy of the essential spectrum is contained inside a circle rather than inside a square of the same surface. This assumption is usually verified in the case of satellite images with non-urban areas.

The image Figure 2a has this characteristic as one can see in its spectrum Figure 2b. We sample this image in a \( \mathbb{Z}^2 \) grid. The new spectrum is duplicated on a \( \mathbb{Z}^2 \) grid (cf. figure 2c). If the Shannon sampling theorem is respected, there is no aliasing. Then, we sample the image on a \( A_2 \) grid. The resulting spectrum is a duplication of the original spectrum on a \( A_2 \) grid (cf. figure 2d).
We observe, with the same density of samples, that the $A_2$ grid allows to detect higher frequencies than using a $Z^2$-grid. In this way, with a fixed maximum frequency, the $A_2$ grid needs about 15% less samples than with the $Z^2$-grid. We denote $D$ the distance between two neighbors in the squared grid (with a 4-connexity neighborhood) and $D_h$ the distance between two neighbors in the hexagonal grid.

Non-hierarchical algorithm have been adapted to a hexagonal sampling [19]. But in order to use the efficiency of hierarchical algorithms, we have to adapt these algorithms to the hexagonal topology.

### 2.2 A hierarchical structure: the hepta-tree

Hierarchical representation of information is common in image processing [3, 2, 13, 7, 18, 15], because the translation of the information at different scales is generally of great interest. On a $Z^2$-grid, we often define a graph corresponding to several levels of resolution for the observed data $Y$ [4, 3] or for the label field $X$ [10, 11, 8].

- Adopting the structure of the quad-tree (cf. figure 3a), a site $s$ has one father denoted $s^-$ and four sons denoted $s^+$. Trees do not have cycles by definition [10], therefore we can introduce non-iterative algorithms [14]. We create a similar structure using a $A_2$ grid at each level of resolution. The structure presented figure 3b allows to obtain a graph, where a site $s$ has one father denoted $s^-$ and seven sons denoted $s^+$. We call this graph a hepta-tree. At the finest level $N$, the number of pixels is $7^N$ ($4^N$ on the quad-tree). Of course, the contour of the image is not squared and follows a fractal shape (cf. figure 5).

Each son of $s$ belongs to a hexagonal shape, and the grid at level $P$ is seven times the reproduction of the grid at level $P - 1$. With this structure using a $A_2$ grid, we are going to adapt existing HMRF algorithms which are based on a $Z^2$-grid.

### 3 UNSUPERVISED SEGMENTATION ON THE HEPTA-TREE

We generalize the unsupervised HMRF algorithm on the quad-tree [8, 9, 15, 17] to the hepta-tree. We suppose that the hierarchical field of label $X$, indexed on the hepta-tree is Markovian in scale, and we suppose that conditional data likelihoods are Gaussian, i.e. $P (Y_s^N | X_s^N) \in \mathcal{N} (Y_s^N, \mu_t, \sigma_t)$. If the finest level is composed of $7^N$ pixels, the hepta-tree has got $N + 1$ levels and each level will be denoted: $X^N = \{X_s^N\}_{s=1}^{7^N}$. The Markovian field $X$ is defined by the a priori distribution $\pi_i = P (X_1^0 = \omega_i)$ ($\omega_i \in \Lambda$, the set of classes) and by the transition probabilities $a_{ij} = P (X_s = \omega_j | X_{s-} = \omega_i)$.

Based on this modeling, many authors [6] proposed image segmentation algorithms including automatic parameter estimation, which steps are the following:

- Initialization of the prior parameters $\{\pi_i, a_{ij}\}$ and conditional parameters $\{\mu_t, \sigma_t\}$. The whole set of parameters is denoted $\phi$.

- For the $k^{th}$ iteration:
  - Upward sweep and Downward sweep on the tree to calculate a posteriori probabilities:
    \[
    \xi_s^{[k]} (i) = P (X_s = \omega_i | Y_s, \phi^{[k-1]}) \tag{1}
    \]
    \[
    \psi_s^{[k]} (i,j) = P (X_s = \omega_j, X_{s-} = \omega_i | Y_s, \phi^{[k-1]})
    \]
  - Updating of prior and conditional parameters using EM, SEM or ICE [8] algorithms.
\* k \leftarrow k + 1. If k < k_{\text{max}} repeat the estimation step using \( g^k \).

- Segmentation step according to the MPM criterion

\[
\hat{X}_s = \arg \max_{\omega_i} P(X_s = \omega_i | Y = y) = \arg \max_{\omega_i} \xi_s(i)
\]  

(2)

The next section presents segmentation results using the two sampling grids.

4 SEGMENTATION RESULTS ON SYNTHETIC IMAGES

We present segmentation results on an image with 4 classes. The noise parameters are on table 1. Two synthetic images are simulated, one on a \( \mathbb{Z}^2 \) grid (cf. figure 4a), an other on an \( A_2 \) grid (cf. figure 5a). The sampling steps are chosen in order to have the same Nyquist frequency, i.e. \( D_h = \sqrt{\frac{2}{3 \pi}} D \approx 1.07D \). The image 4a is composed of \( 128^2 = 16384 \) pixels and the image 5a is composed of \( 7^2 = 16807 \) pixels. Both algorithms give a good segmented map (cf. figures 4b and 5b) with approximately the same error rate (1%) and the parameter estimation is accurate (cf. table 2). Results were obtained using a Gaussian noise, but other laws can be introduced [17, 12].

The interest of the hepta-tree structure on the quadtree in addition to the optimization of the sampling rate (in the case of a circular spectrum), is to require less space memory. Indeed, if we have \( N \) points at the finest level of the tree, the whole quadtree requires \( \frac{1}{2} N \) sites, and the hepta-tree \( \frac{7}{8} N \).

<table>
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<th>2</th>
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<th>4</th>
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<td>100</td>
<td>160</td>
<td>210</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>25</td>
<td>25</td>
<td>35</td>
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Table 1: Gaussian noise parameters of the 4 classes of the synthetic images (figures 4a and 5a).

5 CONCLUSION

This paper has addressed the problem of unsupervised image segmentation over hexagonal grid. We succeeded in extending the hierarchical unsupervised segmentation based on the \( \mathbb{Z}^2 \) grid to the hexagonal grid, with the construction of a new graph called the hepta-tree. This new model is attractive because it allows to work on images sampled in an optimal way. Furthermore, the hepta-tree structure has less levels than the quad-tree structure and it requires less memory space.

References


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<tr>
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Table 2: Parameters estimation on $Z^2$ and $A_2$ grids (EM algorithm). (quad-tree/hepta-tree).

Figure 4: Segmentation using a quad-tree a) Synthetic image of size 128 × 128 pixels b) Segmented image obtained c) Perfect segmented image

Figure 5: Segmentation using hepta-tree a) Synthetic image of size 72 pixels b) Segmented image obtained c) Perfect segmented image